**EQUIVALENCE AND OPTIMIZATION : A CASE OF PARTIAL TRIALLEL CIRCULANT DESIGNS**

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| **Article Summary** (In short - What is your article about – Just 2 or 3 lines) | This research deals with the important question in deciding whether a circulant partial triallel cross (PTC) can be obtained from another PTC by just randomizing (relabelling) the lines. This helps to choose the optimum designs with minimum efforts and supplies the interpretation and solutions to the conjecture of Hinkelmann (1967). Statements and theorems relating to the above-mentioned paper have been critically reviewed and needfully modified. PTC's of size 3N and 4N have been proposed and optimum PTC's for parents N=6 to 20 have also been listed. |
| **Category:** | Genetics |

**Your full article ( between 500 to 5000 words) - -** Do check for grammatical errors or spelling mistakes

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## 1 INTRODUCTION

A partial triallel cross (PTC) is defined as a sample of all the possible three-way crosses to gather information on the combining abilities of the parents (Hinkelmann 1963, 1965). Hinkelmann (1967) further constructed and analysed PTC's using circulant PBIB designs of Kempthorne (1953). Assuming the N parents having been denoted by i = 1,2,...,N, the triplet (i,j)k stands for a typical three-way cross where i and j are half parents and k is full-parent. Here (i,j)k and (j,i)k are taken equivalent in that the two half-parents are symmetrically placed but k being full parent the triplet (i,k)j is different from earlier two triplets from genetic interpretation. The model for the genetic part of such a cross is

$$Y_{(i,j)k} = \mu + h_i + h_j + g_k + \in_{(i,j)k} \qquad (1)$$

(i,j,k = 1,...,N,i≠j≠k) where $g_k$ is called the general effect of the Ist kind of line k and $h_i$ the general effect of the second kind of line i. The $\in_{(i,j)k}$ are assumed to be independent random variables with mean zero and variance $\sigma^2$. For a comprehensive model, estimation and genetical interpretation of variances among crosses etcetera, reference is made to Cockerham (1961), Rawlings and Cockerham (1962) and Hinkelmann (1963,1965,1967); and that for the notions of PBIB designs, association schemes and construction of PTC's to Bose and Mesner(1959), Hinkelmann(1965,1967) and Zoellner and Kempthorne(1954).

In what follows we first review the work of Hinkelmann for the concepts and comprehension of the subject and then modify the notations in accordance with Arya(1983,1989) to make our treatment emendable to algebraical manipulations for establishing the equivalence between two PTC's.

Hinkelmann constructed the circulant partial triallel crosses using circulant (or cyclic) association scheme as basic PBIB applied to two sets, namely

$$\Omega_H = (1_H, 2_H, \ldots, N_H) \text{ and } \Omega_F = (1_F, 2_F, \ldots, N_F) \qquad (2)$$

of lines in their function as half parents and full-parents respectively. The resulting associate classes were represented by 11:t-th and 22:t-th associates for t=0,1,...,M, where M = N/2 for N even or (N-1)/2 for N odd respectively. For 11:t-th, each $i_H$ is considered the 0-th associate of itself and of no other element. Also if $j_H$ is the 11:s-th associate of $i_H$ for some $s(0 \leq s \leq M)$ then $j_F$ is taken to be the 12:s-th associate of $i_H$. In particular, every element $i_F$ is the 12:0-th associate of $i_H$ and of no other element. A similar interpretation holds for 22:t-th associates.

The construction of PTC corresponds to the construction of an incomplete block design with 3-plots having two different treatments from $\Omega_H$ and one from $\Omega_F$ , this one being different from the first two. For balancing property each element from $\Omega_F$ should occur r times and each element from $\Omega_H$ should occur 2r times. It may however, be clarified that from the point of view of design, a PTC and a PBIB with 3-plot blocks are equivalent but they refer to different concepts from analysis and interpretation point of view. In PBIB we have three observations on three treatments per block and inter as well as intra-block analysis is possible, while in PTC there is only one observation per block, pertaining to a three-way-cross plant and only inter-block analysis is possible.

For the parameters of circulant PBIB used above, we note that the number of 11:t-th associates, of 12:t-th associates and 22:t-th associates is equal to the number of t-th associates, namely, $n_t$ =2, for t=1,2,...,M except for t=M if N is even, where $n_M$ =1 and $n_0$ =1. Also if two treatments are 11:t-th associates, then the number of elements common to 12:$s_1$ -th associates of the lst and the 12:$s_2$ -th associates of the second element is equal to $p^t_{s_1,s_2}$ , where $p^t_{s_1,s_2}$ is a parameter of the second kind for the basic PBIB (t,$s_1$,$s_2$=0,1,...,M). For completeness the values of $p^t_{s_1,s_2}$ are given below


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$p^0_{ss'}$ =1 for s=s′=0

=1 for s=s′=M and N even

=2 for s=s′ ((s,s′) $\neq$ (0,0) or (M,M) for N even)

=0 otherwise

$p^t_{ss'}$ =1 for t=s-s′, t=s′-s

t=s+s′, t=N-s-s′ (3)

=0 otherwise

(t=l,2,...,M; except for N even, t=M)

for N even :

$p^M_{ss'}$ =1 for (s,s′)=(M,0) or (0,M)

=2 for M=s+s′ ((s,s′) $\neq$ (M,0),(0,M))

=0 otherwise. (4)

Further, let $\lambda_{11:t}, \lambda_{12:t}, \lambda_{22:t}$ denote the number of times two 11:t-th associates, two 12:t-th associates and two 22:t-th associates occur together in a block, respectively. Then $\lambda_{11:0}$ =2r, is the number of occurrence of a treatment as half-parent, $\lambda_{12:0}$=0, as $i_H$ and $i_F$ cannot occur together in a block, and $\lambda_{22:0}$ =r, is the number a line occurs in the design as a full-parent. Also $\lambda_{22:t}$ =0 (t=1,2,...,M) since each block contains only one element from $\Omega_F$ .

Hinkelmann (1963,1965 and 1967) derived a set of conditions to be satisfied by the parameters of the second kind of PBIB and made use of these to construct the PTC's. He distinguished between a PTC, an elementary PTC and a simple PTC. We shall, however, be using a single term i.e. PTC for all such PTC's for reason to be explained shortly. Briefly stated, according to him, a necessary and sufficient condition for a PTC of size N, PTC(N), to exist was that $t_0$ and $s_0$ ( not 0 and =l,2,...,M) would satisfy,

(a) $t_0 = 2s_0$

or

(b) $t_0 = N-2s_0$ (5)


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and that such a PTC(N) was represented uniquely by the fundamental set

$$S(1+s_0, 1+N-s_0) = \{(1+s_0, 1+N-s_0)1\} \tag{6}$$

from which the whole PTC(N) was generated in a circulant manner having the following crosses

$(1+s_0, 1+N-s_0)1$

$(2+s_0, 2+N-s_0)2$

.

.

$(N+s_0, 2N-s_0)N$ (7)

wherein all numbers greater than N are reduced mod N and zero is replaced by N. It was further shown that a necessary and sufficient condition for a PTC(2N) to exist, the numbers $t_0$, $s_1$ and $s_2$ ($\neq 0$, $s_1 \neq s_2$, $t_0, s_1, s_2 = 1,2,\ldots,M$) satisfy one of the four conditions

(a) $t_0 = s_1 - s_2$

(b) $t_0 = s_2 - s_1$

(c) $t_0 = s_1 + s_2$

(d) $t_0 = N - s_1 - s_2$ (8)

For any triplet ($t_0 : s_1, s_2$) satisfying (8a) or (8b) the corresponding PTC(2N) was generated uniquely by the fundamental sets

$$S(1+s_1, 1+s_2) \text{ and } S(1+N-s_1, 1+N-s_2) \tag{9}$$

If ($t : s_1, s_2$) satisfied (8c) or (8d) the PTC(2N) was generated by

$$S(1+s_1, 1+N-s_2) \text{ and } S(1+s_2, 1+N-s_1) \tag{10}$$

He also obtained among other things, the necessary and sufficient conditions for a PTC(2N), consisting of two PTC(N) (actually this also applies to PTC,s constructed under (9) and (10)) to be connected in his Theorem 1 and dealt with other aspects of balancing and analysing the crosses so developed.

## 2 METHODOLOGY

Let us modify the notations of Hinkelmann to that of Arya(1983,1989). We denote the N lines by i=0,1,...,N-1. Now marking 0-th line as full parent, the remaining N-1 lines could form (N-1)(N-2)/2 single crosses in an arbitrary manner which in combination with 0-th line as full-parent would give rise to a set of (N-1)(N-2)/2 triallel crosses. If, however, we confine to circulant PTC's only, the possible single crosses which could be combined with the 0-th full parent are M(=(N-1)/2) if N is odd or M-1(M=N/2) if N is even. This is because the coefficient matrix A, say, in the normal equations of (1) will be a real symmetric circulant. Denoting the elements in first row of A by $a_i$ (i=0,1,...,N-1), one would observe that

$$a_i = a_{N-i}, \qquad i = 0,1,\ldots,N-1, \tag{11}$$

showing that Line i would combine with Line N-i to form a single cross and to none else. Further each $a_i(i \neq 0)$ is either 0 or 1. In fact the relationship of (11) is a consequence of circulant association scheme with M associate classes, wherein the t-th associates (t=0,1,...,M) of any treatment $\alpha$ (=0,1,...,N-1) is given by

$$C_{\alpha,t} = (\alpha+t, \alpha+N-t) \tag{12}$$

In particular

$$C_{0,t} = (t, N-t) \quad . \tag{13}$$

and, in view of (13), both elements in (11) are always equal(0 or 1). They are not only equal in magnitude but are also equal in their cosine functions, which are employed in analysing these PTC's. For example, cos(2$\pi$)(i)/N = cos(2$\pi$)(N-i)/N for all i. In general for two numbers k and k′, k′=k in their cosine functions if k′=rN$\pm$ k, where r is an integer. To avoid the duplicacies in elements we consider only the M distinct associate classes along with their multiplicities defined earlier. We define a set of integers

$$E_N : 0,1,\ldots,M \tag{14}$$

wherein all numbers greater than M are reduced not only modulo N but also for the equivalence k=N-k (k=0,1,...,M). We say k′=k in $E_N$ if k′=rN$\pm$k. $E_N$ so defined presents a closed set under multiplication. Two Multiplication Tables(MT) are listed in Table-1 for illustration and for use in our later discussion.

Table-1 : Multiplications in elements of $E_N$ for N=12 and 13

---------------------------------------------------------------------------

| | (a) N=12 | (b) N=13 |
|---|---|---|
| | column | column |
| Row | 1 2 3 4 5 6 | 1 2 3 4 5 6 |


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| | | |
|---|---|---|
| 1 | 1 2 3 4 5 6 | 1 2 3 4 5 6 |
| 2 | 2 4 6 4 2 0 | 2 4 6 5 3 1 |
| 3 | 3 6 3 0 3 6 | 3 6 4 1 2 5 |
| 4 | 4 4 0 4 4 0 | 4 5 1 3 6 2 |
| 5 | 5 2 3 4 1 6 | 5 3 2 6 1 4 |
| 6 | 6 0 6 0 6 0 | 6 1 5 2 4 3 |

In present notations the conditions for the existence of a PTC(N) of Hinkelmann contained in (5) can be expressed by a single equation

$$t_0 = 2s_0 \in E_N \qquad (15)$$

for $t_0$ =1,2,...,M and $s_0$ =1,2,...,M for N odd or $s_0$ =1,2,...,M-1 for N even. We denote a PTC with parameters of (15) by C($t_0$ :$s_0$) and the underlying design by D($s_0$). D($s_0$), therefore, characterises the fundamental set {($s_0$ ,N-$s_0$ )0}. Note that on adding unity to each number of this FS we get the set {(1+$s_0$ ,1+N-$s_0$ )1} of Hinkelmann. The number of solutions of (15) are M or M-1 if N is odd or even respectively, and this is the number of possible PTC(N) which could be constructed as pointed out in foregoing paragraph. Thus out of (N-1)(N-2)/2 PTC(N) under complete triallel cross(CTC) only M or M-1 are available with circulant scheme and rest of the crosses get zero probability. We can say that a natural degree of fractionation of complete circulant triallel cross (N) vis-a-vis a CTC is 1/N-1 if N is even or 1/N-2 for N odd.

Coming to PTC(2N), the first and second pairs of equations (8) change respectively to

(a) $t_0 = |s_1 - s_2|$

and (b) $t_0 = s_1 + s_2 \in E_N$ (16)

for $t_0$=1,2,...,M and $s_1 \neq s_2$ =1,2,...,M. Let these PTC's be characterized by C($t_0$ :$s_1$ ,$s_2$) and $C^*$($t_0$ :$s_1$ ,$s_2$) and corresponding designs by D($s_1$ ,$s_2$ ) and $D^*$($s_1$ ,$s_2$ ) respectively. D($s_1$ ,$s_2$ ), then refers to the two FS's {($s_1$,$s_2$)0,(N-$s_1$,N-$s_2$)0}and the $D^*$($s_1$ ,$s_2$ ) to {($s_1$,N-$s_2$)0,($s_2$ ,N-$s_1$ )0}. The total PTC(2N), which equals the number of solutions of (16), is thus M(M-1) for N odd or $(M-1)^2$ for N even. The natural fractionations of these PTC's vis-a-vis CTC being (N-3)/(N-2) or (N-2)/(N-1) if N is odd or even respectively.

The number of PTC(N) along with PTC(2N) constitutes the CTC implying that there is a one-to-one correspondence between the FS's and the PTC's. Therefore a circulant PTC, an elementary PTC or a simple PTC of Hinkelmann are all synonymous so far circulant samples are concerned.

PTC(2N) could alternatively be constructed by combining the PTC(N) in two's, say it PTC(N+N)= $C(t_1 :s_1, t_2:s_2)$, as suggested by Hinkelmann. Thus we get M(M-1)/2 or (M-1)(M-2)/2 combinations according as N is odd or even respectively. This paves the way for constructing PTC(rN) for r>2. A look at (16) suggests that at the most two values of s can be associated with a single value of t. Therefore conditions (2) and (3) of Hinkelmann provide no means to construct PTC's for r>2 with a single parameter t and one has to generate them by combining PTC(N) and/or PTC(2N). With so many PTC's at hand, one would like to select the optimum PTC in terms of some efficiency criterion. We now proceed to deal with this aspect.

## 3. EQUIVALENCE CRITERION

In order to establish the equivalence between two partially balanced designs one fixes some criterion to be qualified by these designs. For example, David(1963,1965), John(1966,1969), and John, Wolock and David(1972) in connection with cyclic paired comparison designs, regard two designs equivalent(or equally efficient) if one can be obtained from the other by a suitable relabelling or permutation of the objects. Here it is not clear which property of the design being studied through relabelling. Moreover with a single labelling several designs qualify to be equivalent. The question arises how to decide the equivalence without disturbing the original labelling. Arya (1989) discussed the equivalence of partial diallel crosses (PDC) using their variance property. In partially balanced incomplete block designs (as is the case with PTC's) the design matrix does play an important role towards the (average) variance used for comparing the performance of treatment contrasts. Hence we associate this quantity to the corresponding design as a criterion of efficiency. Hinkelmann also regards two PTC's as equivalent if they possess the same $\lambda$-parameters. His criterion also leads to the variance property. We shall use the variance criterion (Arya 1983,1989) to make it more consistent since two PTC's having the same $\lambda$-parameters may possess different eigen roots and hence different average variances.

For completeness we define a PDC (Arya 1983) which resembles a PTC(2N) in design and analysis.

Definition 3.1 : A PDC design in which the 0-th line has single crosses with four lines, $s_1$ ,N-$s_1$ ,$s_2$ and N-$s_2$ is denoted by D($s_1$ ,$s_2$ ) where $s_1 \neq s_2 \neq 0$; $s_1$ ,$s_2$ = 1,2,...,M in $E_N$ .

Thus D($s_1$,$s_2$) characterizes a fundamental set of 4N single cross, i.e. {$s_1\times0$, $(N-s_1)\times0$, $s_2\times$



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0, (N-$s_2$)×0 } vis-a-vis a set of 2N three-way-crosses for a similar PTC design, C($t_0$ :$s_1$ ,$s_2$ ). Hence from design point of view D($s_1$ ,$s_2$) and C($t_0$ :$s_1$ ,$s_2$) are similar but they differ in analysis and interpretations. The analysis of PTC becomes more complicated for the inclusion of the parameters $t_0$ and $p^t_{s_1,s_2}$ to be discussed shortly.

The eigen values of the coefficient matrix A, in the least squares matrix of the PDC's are (Arya 1983,1989)

$$\theta_k = \sum_{t=0}^{M} n_t a_t \cos(tk\upsilon) \tag{17}$$

(k = 0,1,...,M) where $n_t$ is the number of t-th associates and $a_t$ is the t-th element in the first row of A, such that $a_0$= 4, $a_t$= 1 when t=$s_1$ or $s_2$ and 0 otherwise, and ν=2π/N. The eigen values of the matrix B (which decides the estimability of h and g-effects for model (1)) in the reduced normal equations for the h-effects under PTC's are (Hinkelmann 1967)

$$\theta_k = \sum_{t=0}^{M} n_t \lambda_t^* \cos(tk\upsilon) \tag{18}$$

where earlier symbols have the same meaning as in (17) and

$$\lambda_t^* = \lambda_{11:t} - (1/2) \sum_{s_1,s_2=0}^{M} \lambda_{12:s_1} \lambda_{12:s_2} p^t_{s_1 s_2} \tag{19}$$

with $P_t$ =( $p^t_{s_1,s_2}$ ) (t=0,1,...,M) as (M+1)×(M+1) association matrices of the parameters of second kind for the circulant PBIB's. It is pointed out that where as λ-parameters under PDC are all non-negative integers (0 or 1), those given by (19) may be negative numbers. Not only that, the λ-values may force the eigen roots as well as the average variance to be negative. A PTC with negative eigen value(s), with a deceptive lower average variance, will not be considered towards optimum designs. For PTC(2N), C($t_0$ ,$s_1$ ,$s_2$ ) under discussion, (19) reduces to

$$\lambda_t^* = \lambda_{11:t} - (1/2)\{p^t_{s_1 s_1} + 2p^t_{s_1 s_2} + p^t_{s_2 s_2}\} \tag{20}$$

For a connected PTC(2N), when B has rank N-1, the average variance(*Av.Var.*) over possible h-differences works to be

$$Av.Var. = 2\sigma^2 (\sum_{k=1}^{M} n_k / \theta_k)/(N-1) \tag{21}$$

The av.var. or any of its derived functions can be used as an index of efficiency for the PTC. For


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example the relative efficiency of plan A compared with plan B can be taken as

$$E_{AB} = \frac{\text{Av. Var. for plan B}}{\text{Av. Var. for plan A}} \tag{22}$$

Since av.var. depends on $\theta$-values through $\lambda$-parameters, the former has a bearance on equivalence of designs which is the subject of next discussion.

The direct way of establishing equivalence between the PTC's, $C(t_0 : s_1, s_2)$ and $C'(t_0' : s_1', s_2')$ or between $C(t_1:s_1, t_2:s_2)$ and $C'(t_1':s_1', t_2':s_2')$ is to check for their $\theta$-values or average variance of the PTC concerned. The $\lambda$-parameters are, for

$C(t_0 : s_1, s_2)$ : $\lambda_{11:0} = 4$, $\lambda_{11:t_0} = 2$, $\lambda_{12:s_1} = 1 = \lambda_{12:s_2}$ and the rest zeros (23a)

$C(t_1:s_1, t_2:s_2)$ : $\lambda_{11:0} = 4$, $\lambda_{11:t_1} = 1 = \lambda_{11:t_2}$, $\lambda_{12:s_1} = 1 = \lambda_{12:s_2}$ and all the rest zeros (23b)

Similar expressions hold for $C'$'s. If for certain combinations of (t,s) the eigen values of (18) through $\lambda^*$'s happen to be equal for C and C′, their equivalence would be established. Since (16) can be satisfied by several sets of values, we expect equivalence to exist.

## 4. SYMMETRY RELATIONS

The equalities in $\theta$-values can be introduced in different ways. It may be either due to symmetry in index values, namely, $(t_1 : s_1)$ could be interchanged with $(t_2:s_2)$ in $C(t_1:s_1, t_2:s_2)$, or it may occur due to equalities in $E_N$ where k′=k if $k' = rN \pm k$ as explained earlier or it may happen due to equalities of p-values where $p^{t}_{s_1,s_2}$ may equal $p^{t'}_{s_1',s_2'}$ even when $t \neq t'$, $s_1 \neq s_1'$ and $s_2 \neq s_2'$. Accounting for such equalities seven types of equivalence are listed in Table-2 for both types of PTC's. The symbol $(t: s) \leftrightarrow (t': s')$ refers to two-way equivalence between the sets $(t_1:s_1)$ and $(t_1':s_1')$ in $E_N$.

Table-2 : Different types of equivalence among the PTC's

(a) PTC designate $C(t_0, s_1, s_2)$

| Given set | Equivalent set | Type |
|---|---|---|



Author: **Dr. Pravin Arya** - Senior Scientist at Indian Agricultural Statistics Research Institute (IASRI), New Delhi

| | | | |
|---|---|---|---|
| $(t_0,s_1,s_2)$ | $\leftrightarrow$ | $(t_0, s_1', s_2')$ | $T_1$ |
| | | $(t_0, s_2', s_1')$ | $T_2$ |
| | | $(t_0, s_2, s_1)$ | $T_3$ |
| | | $(t_0', s_1, s_2)$ | $T_4$ |
| | | $(t_0', s_1', s_2')$ | $T_5$ |
| | | $(t_0', s_2, s_1)$ | $T_6$ |
| | | $(t_0', s_2', s_1')$ | $T_7$ |

(b) PTC designate $C(t_1{:}s_1, t_2{:}s_2)$

| Given set | | Equivalence set | Type |
|---|---|---|---|
| $(t_1 : s_1)(t_2 : s_2)$ | $\leftrightarrow$ | $(t_1' : s_1')(t_2' : s_2')$ | $T_1$ |
| | | $(t_1 : s_1)(t_2' : s_2')$ | $T_2$ |
| | | $(t_1' : s_1')(t_2 : s_2)$ | $T_3$ |
| | | $(t_2 : s_2)(t_1 : s_1)$ | $T_4$ |
| | | $(t_2 : s_2)(t_1' : s_1')$ | $T_5$ |
| | | $(t_2' : s_2')(t_1 : s_1)$ | $T_6$ |
| | | $(t_2' : s_2')(t_1' : s_1')$ | $T_7$ |


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## 5. EQUIVALENCE THEOREMS AND THEIR APPLICATIONS

To establish the sort of relations given in Table-2 we quote some theorems from Arya(1989) whose proofs are simple but avoided here, and may be referred to if so desired.

Theorem 5.1 : Two sets S(i,j,...,q) and S′(i′,j′,...,q′) of equal order are equivalent iff

$$\mu \times \{i,j,...,q\} \rightarrow \{i',j',...,q'\}$$

and simultaneously

$$\lambda \times \{i',j',...,q'\} \rightarrow \{i,j,...,q\}$$

where operators $\mu$ and $\lambda$ both prime to N are inverse to each other, i.e. they satisfy $\mu \times \lambda=1$ in $E_N$ .

Symbolically we may denote such relations by

$$\{i,j,...,q\} \quad < \frac{\mu}{\lambda} > \quad \{i',j',....q'\}$$

or by S $\leftrightarrow$ S′, implying that randomization $\mu$ on S yields S′and randomization $\lambda$ on S′yields S. We shall call such sets as mutually generated (MG), equivalent or randomized sets. Let us term set S from which other sets are generated as Leading Set(LS). A LS thus may represent a family of MG sets. A rule of thumb to decide whether S(i,j,...,q) and S′(i′,j′,...q′) are MG sets is to check S under columns C′(i′,j′,...,q′) and vice-versa of multiplication tables such those given in Table-1.

Theorem 5.1 ensures that two PTC's characterized by S and S′ sample exactly the same crosses though put in a different order. This is expressed as

Theorem 5.2 : Two PTC's characterized by MG sets are equivalent otherwise they are distinct, namely, they have different efficiencies.

It may be pointed out that Theorem 5.1 strictly applies to leading sets of PDC no two elements of which were equal. A PTC, however, is characterized by two or more sub-sets, which may be recognised as extended sets to explain the equivalences in Table-2.

Theorem 5.3 : For N prime, a given leading set S(i,j,...,q) has exactly M equivalent sets given by S = S{k $\times$ (i,j,...,q)}for k = 1,2,...,M.

Example 5.1 : Consider N=7, M=3 and $E_N$ = 0,1,2,3. The two LS's ie., (1:1,2) and (1:1,3) contain 3 MG sets each accounting for the 6 PTC(2N). One may check for the randomizations viz.,


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$$(1{:}1,2) < \frac{2}{3} > (2{:}2,3) < \frac{2}{3} > (3{:}1,3) < \frac{2}{3} > (1{:}1,2)$$

$$(1{:}2,3) < \frac{2}{3} > (2{:}1,3) < \frac{2}{3} > (3{:}1,2) < \frac{2}{3} > (1{:}2,3)$$

For N mixed the leading set will not yield M equivalent sets. Now more symmetries will be introduced and more distinct classes will be formed reducing the frequencies of equivalent PTC's.

Example 5.2 : Let us consider a PTC(2N) as combination of two PTC(N) for N=9. Each of the two leading sets generates only three MG sets and account for all the six PTC's. The randomization being

$$\{1{:}4,2{:}1\} < \frac{4}{2} > \{1{:}4,4{:}2\} < \frac{4}{2} > \{2{:}1,4{:}2\} < \frac{4}{2} > \{1{:}4,2{:}1\}$$

$$\{1{:}4,3{:}3\} < \frac{2}{4} > \{2{:}1,3{:}3\} < \frac{2}{4} > \{3{:}3,4{:}2\} < \frac{2}{4} > \{1{:}4,3{:}3\}$$

Looking at the randomization carried in example 5.1 and 5.2, it will be observed that a single $\lambda$ could operate on both the sub-sets and were of the types $T_5$ or $T_7$ and $T_1$ or $T_7$ in Table-2(a) and Table-2(b) respectively.

When N is even and multi factor, due to non-existence of pairs ($\mu,\lambda$) of Theorem 5.1 the necessary condition of theorem will no longer hold, though sufficient one still does. More equalities in $p^{t}_{s_1,s_2}$ are introduced and distinct classes increase at the cost of equivalent ones. The equivalence type in Table-2 will be confined to sub-sets only, recall that only a prime $\lambda$ operates on the total set.

Example 5.3 : Consider a PTC($t_0$ :$s_1$ ,$s_2$) for N=8. Since 3 is only prime to 8, a leading set can generate at the most 4 MG sets. Referring to equivalencies in Table-2(a), the 9 PTC,s fall into 5 distinct groups, namely,

$$\text{(i):}(1{:}1,2) \xleftrightarrow{T_1} (1{:}2,3) \xleftrightarrow{T_5} (3{:}1,2) \xleftrightarrow{T_1} (3{:}2,3),$$

(ii):(2:1,3), (iii):(1:3,4) $\leftrightarrow$ (3:1,4), (iv):(2:2,4) and (v):(4:1,3).

First and second group form singular samples having 1and 2 roots (other than $\theta_0$ ) zero, the last three groups yield 1 root negative.

It is found that $\lambda$-parameters for C(2:1,3) and C(4:1,3) are the same but their $\theta$-values


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are different. Therefore, equalities of $\lambda$-parameters does not necessarily imply equalities of eigen values. Such more exceptions may be cited.

6. CONSTRUCTION OF PTC's :

A systematic way of constructing the entire set of PTC's is discussed below:

(i) PTC(N)

A primitive solution of (15), ($t_0$ :$s_0$ ) = (2:1), is used as generating set(GS) from which the entire FS's of PTC(N), i.e.,

$$S_k = \{k \times (2:1)\} \rightarrow \{2k:k\} \qquad (24)$$

for k through 1 to M (N odd) or M-1 (N even) is obtained. If N is prime all these sets are MG sets and we obtain so many structurally equivalent PTC(N). If N is a mixed number, the number of distinct PTC(N) increases due to symmetries and (24) does not yield the complete set. One has to search the remaining solutions directly (see example 5.2).

(ii) PTC(2N)

A primitive solution (1:1,2) of (16) is selected as generating set. A set of M-1 PTC's is raised as

$$S_k = \{1:k,1+k\}, k=1,2,...,M-1 \qquad (25)$$

If N is prime, the above M-1 sets are all distinct and each one gives rise to M equivalent FS's through randomisation in $E_N$ , namely,

$$S_{k,l} = \{l \times (1:k,1+k)\} \ (l=1,2,...,M) \qquad (26)$$

Hence (26) yields all the possible M(M-1) PTC's. If N is not prime, the M-1 sets in (25) would not generate the entire M(M-1) sets due to factorability of N and consequently one has to search some more distinct sets as solutions of (26) as pointed out earlier. For illustration, with N=9, (25) yields only 3 tenable sets from each $S_1$, $S_2$ and $S_3$ accounting for 9 PTC's only. One more solution of (16), ie., $S_4$ =(3:1,3) having 3 equivalent sets makes up the full set of 12 PTC's. If N is a multifactor 12, say, with 5 as prime and with only 25 PTC's there would be 16 distinct solutions of (16). Out of these except C(4:2,6) all the rest are either singular or have negative roots, an undesirable property sending a wrong signal for its low value.

(iii). PTC(N+N)

In order to construct a PTC(2N) as combination of two PTC(N), the straight way is to combine all PTC's in two's and thus get ${}^{M}C_2$ or ${}^{(M-1)}C_2$ PTC(2N) if N is odd or even respectively. One has to set up the equivalence thereafter. Alternatively one can use the primitive set (2:1) to produce q(< M/2) distinct sets

$$S_\lambda = \{ (2:1),(2\lambda:\lambda) \},\ \lambda=2,3,...,q \qquad (27)$$

where each $\lambda$ possesses its inverse element $\mu \in E_N$ . With N prime, there are M/2 or (M-1)/2 pairs of $(\lambda,\mu)$ as M is even or odd respectively. Further when M is even one pair $(\lambda,\mu)$ has $\lambda=\mu$ and it generates only M/2 MG sets while for $\lambda \neq \mu$ , each of the (M-1)/2 pairs $(\lambda,\mu)$ generates exactly M equivalent designs $S_{\lambda,k}$ (k=1,2,...,M). There exist thus M(M-1)/2 total PTC's.

Example 6.1 : N=11. We have (2,5) and (3,4) as $(\lambda,\mu)$ pairs. Using (27) one gets the distinct PTC's as

$$S_2 = \{ (2:1),(4:2) \}$$

$$S_3 = \{ (2:1),(5:3) \} \qquad (28)$$

From each of these two sets one gets five MG fundamental sets, $S_{2,k}$ and $S_{3,k}$ (k=1,2,...,5) thus accounting for all the 10 PTC's.

Example 6.2 : Let N=13. Here (2,6), (3,4) and (5,5) are the $(\lambda,\mu)$-pairs. They yield $S_2 =\{(2:1),(4:2)\}$, $S_3 =\{(2:1),(6:3)\}$ and $S_5 =\{(2:1),(3:5)\}$ as distinct sets. Notice that $S_5$ would only account for 3 MG sets, ie.,

$$S_{5,1} = \{(2:1),(3:5)\} = S_{5,5}$$

$$S_{5,2} = \{(4:2),(6:3)\} = S_{5,3}$$

$$S_{5,4} = \{(5:4),(1:6)\} = S_{5,6}$$

The above three leading sets thus generate in all 15 PTC's.

(iv). PTC's of higher orders :

PTC's of order 3N or 4N may be obtained as combinations from those of lower orders. For example, a PTC(3N) can be obtained either by combining 3 PTC(N) as C{($t_1$ :$s_1$),($t_2$ :$s_2$),($t_3$


Author: **Dr. Pravin Arya** - Senior Scientist at Indian Agricultural Statistics Research Institute (IASRI), New Delhi

$:s_3)\}$or one PTC(2N) with one PTC(N) as $C\{(t_0 :s_1 ,s_2),(t_3 :s_3)\}$. Similarly a PTC(4N) can consist either of four PTC(N) or a combination of two PTC(2N).or else a combination of two PTC(N) and one PTC(2N). One, therefore, has a scope to select the best out of a range of comparable PTC's based on the criterion fixed. The theorems and discussion on equivalence holds good for all these PTC's except that more symmetries will be introduced through subsets of Table-2 and perhaps beyond that. For example Table-2 fails to explain the equivalence of the two PTC4N for N=11, i.e. C{(1:2,3),(3:1,4)} and C{(1:3,4),(4:1,3)}with an equal average variance of $0.5146\sigma^2$ .

## 7. CONNECTED PTC's

Hinkelmann defined a PTC to be connected if all differences $h_i - h_{i'}$ (and consequently all $g_i - g_{i'}$ , $(i,i'=0,1,...,N-1)$ were estimable. According to his Theorem-1, the necessary and sufficient conditions for a $PTC(N+N)=C(t_1 :s_1 ,t_2 :s_2)$ to be connected is that $(s_1 +s_2 ,N)=1$ and $(|s_1 -s_2|,N)=1$, where $d=(a,b)$ is the greatest common deviser of a and b.

This theorem truly applies to a PDC, the $D(s_1 ,s_2)$, discussed in S 5 having eigen values of (17) and not to a PTC with eigen values of (18) which utilizes $p^t_{s_1,s_2}$ 's through $\lambda^*$-parameters. The observation of Arya(1983) that "odd line samples of PDC were all singular for even N, was a verbal statement of this theorem (also see Curnow 1963). To substantiate our statement some PTC's in violation of the Theorem of Hinkelmann are listed in Table-3.

Table -3 PTC's showing counter examples to theorem on connectedness of Hinkelmann

| N | PTC's which should have been singular but aren't | PTC's which should not be singular but are |
|---|---|---|
| 8 | (4:1,3), (2:2,4) | $(1:1,2) \rightarrow (1:2,3) \rightarrow (3:1,2) \rightarrow (3:2,3)$, (2:1,3) |
| 12 | (2:4,6), (4:2,6) | $(1:2,3) \rightarrow (5:2,4)$, |
| | $(6:2,4) \xleftrightarrow{P} (6:1,5)^*$ | $(1:3,4) \rightarrow (5:3,4)$, |


Author: **Dr. Pravin Arya** - Senior Scientist at Indian Agricultural Statistics Research Institute (IASRI), New Delhi

| | | |
|---|---|---|
| | $(2:1)(6:3) \rightarrow (2:5)(6:3)$ | $(2:1)(4:4) \rightarrow (2:5)(4:4)$, |
| | | $(4:2)(6:3) \xleftrightarrow{P} (4:4)(6:3)$, |
| | | (2:5)(4:2) |
| | $(2:6,8) \rightarrow (6:2,8)$ | (1:1,2), (1:2,3), (1:3,4), |
| | | (1:4,5), (1:5,6), (!:6,7) |
| 16 | $(8:1,7) \rightarrow (8:3,5)$, | and twelve of their generated sets |
| | $(4:4,8) \rightarrow (8:2,6)$, | |
| | $(4:2)(8:4) \rightarrow (4:6)(8:4)$ | |

-----------------------------------------------------------------------------------------------------

* The equivalence $\xleftrightarrow{P}$ refers to equalities through p-values, rather than the MG equivalence.

8. BALANCED PTC's :

According to Hinkelmann a PTC is balanced if every combination ($i_H$ ,$j_H$) and every combination ($i_H$ ,$k_H$ ) occurs exactly q times for $1\leq q\leq N-2$ which is possible if N were odd.

For q=1, balanced set of PTC(N),C($t_0$ ,$s_0$) consists of the M sets $S_K$ contained in (24) wherein each $t_0$ and each $s_0$ ($1\leq t_0 ,s_0 \leq M$) occurs exactly once.

For q=2 a set of PTC($t_0$ ;$s_1$ ,$s_2$ ) is balanced if every $t_0$($1\leq t_0 \leq M$) occurs once and every s($1\leq s\leq M$) occurs exactly twice. For N prime, M-1 such balanced sets are always available as $S_k$ (for k=1,2,...,M-1) of (25), each containing M PTC's contained in (26). A more balanced and appealing PTC is worth considering in which all pairs ($s_1$ ,$s_2$) attain a balance. One such PTC, say it half-fraction CTC can be raised using (M-1)/2 (integer) balanced sets in which every $t_0$ occurs (M-1)/2 times and every pair ($s_1$ ,$s_2$) occurs once in the whole design.

Example 8.1 : Suppose N=11. Using (25) and (26) we get 4 balanced sets of 5 equivalent PTC's as :

$S_1$ :(1:1,2),(2:2,4),(3:3,5),(4:4,3),(5:5,1)



Author: **Dr. Pravin Arya** - Senior Scientist at Indian Agricultural Statistics Research Institute (IASRI), New Delhi

$S_2$ :(l:2,3),(2:4,5),(3:5,2),(4:3,1),(5:1,4)

$S_3$ :(1:3,4),(2:5,3),(3:2,1),(4:1,5),(5:4,2)

$S_4$ :(1:4,5),(2:3,1),(3:1,4),(4:5,2),(5:2,3)

Note that ($S_1$ ,$S_2$) and ($S_3$ ,$S_4$) form two half-fraction CTC's. In Example 4 of Hinkelmann, the four balanced sets were arbitrarily written and hence such a balance was not obvious.

9. OPTIMUM DESIGNS FOR PTC's

In order to evaluate the parents in hybrid combinations (PTC's here) one should naturally go for the optimum plan. Such plans have been listed in Appendix A for N=6 to 20 for PTC's of size 2N, 3N and 4N along with their index of efficiency, namely, the average variance(Av. Var). Hopefully these will meet the need of a breeder desiring to evaluate his material for its three-way cross potential. The explanation for the code of designs is recapitulated here for ease of the reader.

A PTCN refers to a PTC of size N, namely, there are N three-way crosses and each parent occurs once as a full-parent and twice as a half-parent. It is denoted by C($t_1$ :$s_1$) with a fundamental set of crosses(in Hinkelmann's simple notation) as {($s_1$+1,N-$s_1$+1)1}.

A PTC2N refers to a PTC of size 2N and is characterized either by C($t_0$ :$s_1$ ,$s_2$) if $t_0$ = | $s_1$ - $s_2$ | or $t_0$ = $s_1$ +$s_2$ $\in E_N$ with the corresponding FS's as {($s_1$+1,$s_2$+1)1, (N-$s_1$ +1,N-$s_2$ +1)1} and {($s_1$+1,N-$s_2$+1)1,$s_2$+1,N-$s_1$+1)1} respectively. Alternatively a combination of two PTC's can result in a PTC2N, C($t_1$:$s_1$,$t_2$:$s_2$) having the F.S. as {($s_1$+1,N-$s_1$+1)1, ($s_2$+1,N-$s_2$+1)}.

Similarly a PTC3N generates a three-way crosses and may consist either as combination of 3 PTCN, namely, C($t_1$:$s_1$,$t_2$:$s_2$,$t_3$:$s_3$) or as combination of a PTC2N and a PTCN, namely, C($t_0$:$s_1$,$s_2$; $t_3$,$s_3$ ) or $C^*$ ($t_0$:$s_1$,$s_2$; $t_3$,$s_3$) as the case may be. Their F.S.'s can be written on the above lines.

Lastly a PTC4N can be obtained by combining either four PTCN or two PTC2N or else one PTC2n and two PTCN and raising the crosses with the help of F.S. The analysis and generic interpretation of all PTC's follow as given in Hinkelmann (1967).

## APPENDIX A

Optimum plans for PTCs of different orders along with their average variances in $\sigma^2$ units for


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N = 6 to 20

| N | PTC2N | Avg. Var. | PTC3N | Avg. Var. | PTC4N | Avg. Var. |
|---|---|---|---|---|---|---|
| 6 | (3:1,2) | 0.4333 | (1:2,3)(2,1) | 0.5290 | -- | -- |
| 7 | (1:2,3) | 0.5675 | (1,3)(2,1)(3,2) | 0.4285 | -- | -- |
| 8 | (2:1,3) | 0.6190 | (4:1,3)(4,2) | 0.4927 | -- | -- |
| 9 | (1:3,4) | 0.5673 | (1,4)(2,1)(3,3) | 0.5175 | (1,4)(2,1)(3,3)(4,2) | 0.2963 |
| 10 | (1:4,5) | 0.5000 | (5:1,4)(4,2) | 0.5237 | (2,1)(2,4)(4,2)(4,3) | 0.3398 |
| 11 | (1:4,3) | 0.5729 | (1:2,3)(2,1) | 0.5363 | (1,5)(2,1)(3,4)(4,2) | 0.3239 |
| 12 | (4:2,6) | 0.5246 | (2,1)(4,2)(6,3) | 0.5100 | (2,1)(2,5)(4,2)(6,3) | 0.3277 |
| 13 | (1:5,6) | 0.5780 | (1,6)(3,5)(4,2) | 0.5000 | (1,6)(2,1)(3,5)(5,4) | 0.3190 |
| 14 | (7:1,6) | 0.5769 | (7:1,6)(4,2) | 0.5156 | (2,1)(2,6)(4,2)(6,4) | 0.3309 |
| 15 | (1:6,7) | 0.5820 | (1,7)(3,6)(6,3) | 0.5106 | (1,7)(3,6)(4,2)(5,5) | 0.3315 |
| 16 | (2:1,3) | 0.5936 | (8:1,7)(4,2) | 0.5261 | (2,1)(4,2)(6,3)(8,4) | 0.3538 |
| 17 | (1:7,8) | 0.5853 | (1:2,3)(2,1) | 0.5507 | (1,8)(2,1)(3,7)(8,4) | 0.3406 |
| 18 | (6:3,9) | 0.5210 | (3:6,9)(6,3) | 0.5210 | (2,1)(4,2)(6,3)(8,5) | 0.3496 |
| 19 | (1:8,9) | 0.5879 | (1:2,3)(2,1) | 0.5546 | (1,9)(2,1)(3,8)(9,5) | 0.3499 |
| 20 | (2:1,3) | 0.5935 | (4,2)(8,6)(10,5) | 0.5126 | (2,1)(4,2)(6,3)(10,5) | 0.3496 |

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